Advanced ODE-Lecture 15
Advanced Lyapunov Theory for Time-varying Systems

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Motivation

1. For uniformly AS of time-varying systems, Theorem 14.2 guarantees uniform AS if Lyaponov condition is satisfied:

$$W_1(x) \le V(t, x) \le W_2(x);$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x),$$

where $W_j(x)$ ($j = 1, 2, 3$) are all positive definite. If $W_3(x) \ge 0$ is semi-positive definite, then, uniform stability can be guaranteed. How about for uniform AS like LaSalle invariance principle? In general, it is not true. However, we have Invariance-Like Theorem.

2. Barbalet lemma plays a key role in Lyapunov stability of time-varying systems. However, there is some certain puzzle unclear.

Barbalat Lemma

Still consider the time-varying system

$$\dot{x} = f(t, x), \tag{15.1}$$

where f is continuous and locally Lip. in x on $[0, \infty) \times D$, and $D \subset R^n$ is a domain containing the origin, and $f(t, 0) \equiv 0$, $\forall\ t \geq 0$.

Barbalat Lemma Let $\varphi : R \to R$ be a uniformly continuous function on $[0, \infty)$. Suppose that $\lim_{t \to \infty} \int_0^t \varphi(s)\,ds$ exists and finite. Then, $\varphi(t) \to 0$ as $t \to \infty$.

Remark 15.1 How to know that a continuous function is uniformly continuous over $[0, \infty)$ is a key in applications. The proof of Barbalat lemma itself is simple. It can be founded in any book of advanced calculus.

Remark 15.2 Barbalat Lemma itself is nothing with (15.1). However, it plays important role but having some puzzle for UAS of (15.1) when it connects trajectories of (15.1). Barbalat Lemma has several variations. I hope you do a survey on Barbalat Lemma if you are interested in this topic.

Invariance-Like Theorem

Theorem 15.1 (Invariance-Like Theorem)

Suppose that $f(t,x)$ is continuous and local Lipschitz with $\| f(t,x) \| \le K$ for all $(t,x) \in [0,\infty) \times D$. Let $V:[0,\infty)\times D \to R$ be of C^1 such that

$$W_1(x) \le V(t,x) \le W_2(x)\,; \qquad (15.2)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -W_3(x),\ \ \forall\, t \ge 0,\ \ \forall\, x \in D, \qquad (15.3)$$

where $W_j(x) > 0$ ($j = 1, 2$) are positively definite and $W_3(x) \ge 0$ is positively semi-definite on D. Then, all trajectories $x(t; t_0, x_0)$ of the system (15.1) with $x_0 \in \{x \in B_r \subset D \mid W_2(x) \le \rho\}$, where $\rho < \min_{\|x\| = r} W_1(x)$, are bounded and satisfy

$$\lim_{t\to\infty} W_3(x(t;t_0,x_0)) = 0. \tag{15.4}$$

Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, the statement is true for all $x_0 \in R^n$.

Proof. Similar to the proof of Theorem 14.1, it can be shown that

$$x_0 \in \{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho} \quad \Rightarrow$$

$$x(t;t_0,x_0) \in \Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \le \rho\} \subset B_r, \text{ for all } t \ge 0.$$

Hence, $\|x(t;t_0,x_0)\| \le r$ for all $t \ge 0$. Since $V'(t,x) \le 0$, $V(t,x)$ is non-increasing on $t \ge t_0$ along the trajectories $x(t;t_0,x_0)$ and bounded below by zero.

Therefore,

$$\lim_{t\to\infty} V(t, x(t; t_0, x_0))$$

exists and finite for each $(t_0, x_0) \in [0, \infty) \times \{x \in B_r \mid W_2(x) \le \rho\}$. Now,

$$\int_{t_0}^{t} W_3(x(s;t_0,x_0))ds \le -\int_{t_0}^{t} V'(s,x(s;t_0,x_0))\, ds = V(t_0,x_0) \le W_2(x_0)$$

holds uniformly in $t_0 \ge 0$, which implies that

$$\lim_{t\to\infty}\int_{t_0}^{t} W_3(x(s;t_0,x_0))ds = W_2(x_0) < \infty, \text{ uniformly in } t_0 \ge 0.$$

Next, we need to prove that $W_3(x(t;t_0,x_0))$ is uniformly continuous in t on $[t_0, \infty)$. For any $t_1, t_2 \ge t_0$, we have

$$\| x(t_2;t_0,x_0) - x(t_1;t_0,x_0) \| \le \int_{t_1}^{t_2} \| f(s, x(s; t_0, x_0)) \| \, ds \le K \,| t_2 - t_1 | < \varepsilon$$

whenever $|t_2 - t_1| < \delta = \dfrac{\varepsilon}{K}$. Therefore, $x(t;t_0,x_0)$ is uniformly continuous on $[t_0,\infty)$. So is $W_3(x(t;t_0,x_0))$ because $W_3(x)$ is continuous on the compact set B_r and $x_0 \in \{x \in B_r \mid W_2(x) \le \rho\} \Rightarrow x(t;t_0,x_0) \in B_r$, so it is uniformly continuous on B_r.

Then, the application of Barbalat Lemma yields

$$\lim_{t\to\infty} W_3(x(t;t_0,x_0)) = 0, \text{ for } x_0 \in \{x \in B_r \mid W_2(x) \le \rho\}, \qquad (15.5)$$

uniformly in $t_0 \ge 0$. (?)

If all the assumptions hold globally and $W_1(x)$ is radially unbounded, for any given $x_0 \in R^n$, there exists $\rho > 0$ s.t.

$$x_0 \in \{x \in B_r \mid W_2(x) \le \rho\} \subset \Omega_{t,\rho}.$$

The rest of the proof is the same to the local. This completes the proof. □

Remark 15.3 There is a puzzle for (15.5). Is it really uniformly convergent with respect to the initial time $t_0 \geq 0$? This is not sure from the above proof through Barbalat Lemma. How to give a rigorous proof is interesting!! Or give a counter-example to show that satisfying Barbalat Lemma in compound function form with $x(t;t_0,x_0)$ is not uniformly convergent!! Please pay attention on this issue because it is fundamental in analysis of time-varying systems.

Remark 15.4 $\lim_{t\to\infty} W_3(x(t;t_0,x_0)) = 0 \quad \Leftrightarrow \quad x(t;t_0,x_0) \to S = \{x \in B_r \mid W_3(x) = 0\}$ as $t \to \infty$. Since S may have no invariant set as for LaSalle Invariance Principle! It is essentially different from the case of autonomous systems.

Remark 15.5 If (15.5) is true, uniformly in $t_0 \geq 0$, then Theorem 15.1 becomes Theorem 14.2 if $W_3(x) > 0$. If $W_3(x) \geq 0$, what additional condition should we impose on S or some others else to conclude uniformly AS? It needs further research!!!

Remark 15.6 Is the following statement true? Why? Show it if yes! Give a counter example if no!

$$\lim_{t\to\infty}\int_{t_0}^{t} W_3(x(s;t_0,x_0))ds = W_2(x_0) < \infty, \text{ uniformly in } t_0 \geq 0 \quad \Rightarrow$$

$$\lim_{t\to\infty} W_3(x(t;t_0,x_0)) = 0, \text{ for } x_0 \in \{x \in B_r \mid W_2(x) \leq \rho\}, \text{ uniformly in } t_0 \geq 0.$$

Remark 15.7 Can the condition of $\| f(t,x) \| \leq K$ be removed or replaced by a weaker condition? I think it could be!

Theorem 15.2 Let $V:[0,\infty)\times G\to R$ be of C^1 such that

$$W_1(x)\le V(t,x)\le W_2(x)\,; \tag{15.6}$$

$$V'(t,x)=\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\le 0,\ \ \forall\, t\ge 0,\ \ \forall\, x\in G\,; \tag{15.7}$$

$$V(t+\eta, x(t+\eta; t, x))-V(t,x)\le -\lambda V(t,x),\ \ 0<\lambda<1, \tag{15.8}$$

where $W_j(x)>0\,(j=1,2)$, $\eta>0$, $x(t+\eta;t,x)$ is the solution of (15.1) that starts at (t,x). For any $x_0\in\{x\in B_r\subset G\,|\,W_2(x)\le\rho\}$, $\rho<\min\limits_{\|x\|=r}W_1(x)$, (15.1) is uniformly AS. Moreover, if all the assumptions hold globally and $W_1(x)$ is radially unbounded, then, (15.1) is globally uniformly AS. If $W_1(x)$ and $W_2(x)$ satisfy

$$W_1(x)\ge k_1\|x\|^c,\ \ W_2(x)\le k_2\|x\|^c,\ \ k_j>0\,(j=1,2),\ \ c>0,$$

then, (15.1) is exponentially stable.

Proof. Choose $r > 0$ and $\rho > 0$ such that $B_r \subset G$ and $\rho < \min_{\|x\|=r} W_1(x)$, and

$$x_0 \in \{x \in B_r \mid W_2(x) \le \rho\} \;\Rightarrow\; x(t;t_0,x_0) \in \Omega_{t,\rho} \subset \{x \in B_r \mid W_1(x) \le \rho\}$$

Based on (15.8), for $t \ge t_0 \ge 0$, $x_0 \in \{x \in B_r \mid W_2(x) \le \rho\}$, we have

$$V(t+\eta, x(t+\eta)) \le V(t, x(t)) - \lambda V(t, x(t)) = (1-\lambda)V(t, x(t)).$$

Since $V'(t,x) \le 0$,

$$V(s, x(s)) \le V(t, x(t)), \quad \forall\, s \in [t, t+\eta].$$

Let $N > 0$ be the smallest positive integer such that $t \le t_0 + N\eta$. Divide the interval

$[t_0, t_0 + (N-1)\eta]$ into $N-1$ equal subintervals of length η each. Then

$$
\begin{aligned}
V(t, x(t)) &\le V(t_0 + (N-1)\eta, x(t_0 + (N-1)\eta)) \\
&\le (1-\lambda)V(t_0 + (N-2)\eta, x(t_0 + (N-2)\eta)) \\
&\quad \vdots \\
&\le \quad \cdots \\
&\le (1-\lambda)^{N-1} V(t_0, x(t_0)) = (1-\lambda)^{N-1} V(t_0, x_0) \\
&\le \frac{1}{1-\lambda}(1-\lambda)^{N} V(t_0, x_0) \le \frac{1}{1-\lambda}(1-\lambda)^{\frac{t-t_0}{\eta}} V(t_0, x_0) \\
&\le \frac{1}{1-\lambda}(1-\lambda)^{\frac{t-t_0}{\eta}} W_2(x_0) .
\end{aligned}
$$

Since $W_1(x) > 0$ and $W_2(x) > 0$, There exist $\alpha_1, \alpha_2 \in K$ such that

$$
\alpha_1(\| x \|) \le W_1(x), \quad W_2(x) \le \alpha_2(\| x \|) .
$$

Let $(1-\lambda)^{\frac{1}{\eta}} = e^{-b}$, which results in $b = \frac{1}{\eta}\ln\frac{1}{1-\lambda} > 0$. So

$$V(t, x(t)) \le \frac{1}{1-\lambda}(1-\lambda)^{\frac{t-t_0}{\eta}}\alpha_2(\| x_0 \|) \le \beta(\alpha_2(\| x_0 \|), t-t_0),$$

where $\beta(r, t) = \frac{r}{1-\lambda}e^{-bt} \in KL$. Then

$$\| x(t; t_0, x_0) \| = \| x(t) \| \le \alpha_1^{-1}\{\beta(\alpha_2(\| x_0 \|), t-t_0)\} = \tilde{\beta}(\| x_0 \|, t-t_0) \in KL.$$

Therefore, (15.1) is uniformly AS. It is easy to show the remaining parts for global case under the given conditions and exponential stability. □

Remark 15.8 If $\lambda \ge 1$ in (15.8), the result of Theorem 15.2 is still true because we can obtain

$$V(t+\eta, x(t+\eta; t, x)) - V(t, x) \le -\lambda V(t, x) \le -\bar{\lambda} V(t, x),$$

where $0 < \bar{\lambda} < 1$. So the restriction of $0 < \lambda < 1$ can be replaced by $\lambda > 0$.

Summary

1. Uniform AS of nonlinear time-varying systems with weak form Lyapunov condition is a very important issue in control theory. It is still a hot research topic in international control circle. You are encouraged to jump into study if you are interested in this issue.

2. The condition (15.8) of Theorem 15.2 has a clear control background. The details are founded in p. 322 of Nonlinear Systems 3rd. authored by H. Khalil.